

# Classification of Roberts actions of strongly amenable $C^*$ -tensor categories on the injective factor of type $III_1$

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## Abstract

In this paper, we generalize Izumi's result on uniqueness of realization of finite  $C^*$ -tensor categories in the endomorphism category of the injective factor of type  $III_1$  for finitely generated strongly amenable  $C^*$ -tensor categories by applying Popa's classification theorem of strongly amenable subfactors of type  $III_1$ .

## 1 Introduction

Since V. F. R. Jones initiated the theory of index for subfactors [10], the theory of subfactors has been developed surprisingly by involving many area of mathematics, e.g., knot theory, low dimensional topology, mathematical physics. One of important points is the connection between subfactor theory and tensor category theory. If a subfactor is given, then a  $C^*$ -tensor category naturally arises as a bimodule category in type  $II_1$  setting, and endomorphism category in type  $III$  setting. These  $C^*$ -tensor categories contain rich informations about the structure of subfactors. See [2] for details of relation between subfactor theory and tensor categories.

Conversely any amenable  $C^*$ -tensor category can be realized as a full subcategory of a bimodule category of an AFD type  $II_1$  factor by the result of Hayashi-Yamagami [4]. (Even if one drops the assumption on amenability, one can still realize it as a bimodule category of some non-AFD type  $II_1$  factor [27].) By tensor product trick, one can realize an amenable  $C^*$ -tensor category as a full subcategory of endomorphisms of an injective type  $III_1$  factor. So natural question is the uniqueness of realization, or more generally how  $C^*$ -tensor functors between two  $C^*$ -tensor categories are realized in the setting of operator algebras.

In [9, Theorem 2.2], Izumi showed that the equivalence of finite  $C^*$ -tensor categories is given by an isomorphism of injective factors of type  $III_1$ , which implies the essential uniqueness of embedding of a given finite  $C^*$ -tensor category into a  $C^*$ -tensor category of endomorphisms of the injective factor of type  $III_1$ . This result can be regarded as the classification of “actions” of  $C^*$ -tensor categories. For example, if a  $C^*$ -tensor category

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comes from a finite group, then his result means the uniqueness of free finite group actions on the injective factor of type III<sub>1</sub>.

Izumi's proof is based on Popa's deep classification result of amenable subfactors [22], [21], [23]. It is well-known that classification of subfactors yields that of discrete amenable group actions on injective factors by considering locally trivial subfactors [22, §5.1.5]. However, one must look carefully on behavior on standard invariants to obtain sufficient classification result. In fact, one get only outer conjugacy of actions from an isomorphism of two locally trivial subfactors. In [9, Theorem 2.2], Izumi uses variants of locally trivial subfactors for finite C\*-tensor categories. To obtain sufficient results, he chose an isomorphism of subfactors which preserves a given tunnel with finite length, and investigated standard invariants carefully. However his method is valid only for finite C\*-tensor categories.

In this paper, we generalize Izumi's result, and show a similar result for finitely generated strongly amenable C\*-tensor categories. Our main tools are the notion of Loi invariant [13] and technique used in [14]. By applying Popa's classification of subfactor of type III<sub>1</sub> and investigating an isomorphism of standard invariants carefully, we deduce the main result.

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## 2 Loi invariant

Our standard references are [2] for subfactor theory, [6] for sector theory and [20] for C\*-tensor categories.

In this section, we recall basic facts on the Loi invariant [13].

Let  $\mathcal{N} \subset \mathcal{M}$  and  $\mathcal{Q} \subset \mathcal{P}$  be isomorphic subfactors with finite index, and  $\alpha: \mathcal{N} \subset \mathcal{M} \rightarrow \mathcal{Q} \subset \mathcal{P}$  be an isomorphism. Fix tunnels

$$\mathcal{M} \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \mathcal{N}_3 \supset \cdots, \quad \mathcal{P} \supset \mathcal{Q}_1 \supset \mathcal{Q}_2 \supset \mathcal{Q}_3 \supset \cdots.$$

Let  $e_k^{\mathcal{N}} \in \mathcal{N}_k$  (resp.  $e_k^{\mathcal{Q}} \in \mathcal{Q}_k$ ) be a Jones projection for  $\mathcal{N}_{k+2} \subset \mathcal{N}_{k+1}$  (resp.  $\mathcal{Q}_{k+2} \subset \mathcal{Q}_{k+1}$ ). Here  $\mathcal{N}_0 = \mathcal{M}$ ,  $\mathcal{Q}_0 = \mathcal{P}$ . Then for any  $k \geq 2$ , we can choose a unitary  $u \in \mathcal{Q}$  such that  $\text{Ad } u \circ \alpha(e_l^{\mathcal{N}}) = e_l^{\mathcal{Q}}$ ,  $0 \leq l \leq k-2$ . Hence  $\text{Ad } u \circ \alpha$  preserves a tunnel

$$\mathcal{M} \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \mathcal{N}_3 \cdots \supset \mathcal{N}_k,$$

i.e.,  $\text{Ad } u \circ \alpha(\mathcal{N}_l) = \mathcal{P}_l$ ,  $0 \leq l \leq k$ . Thus  $\text{Ad } u \circ \alpha$  induces an isomorphism  $\mathcal{N}'_l \cap \mathcal{M} \rightarrow \mathcal{Q}'_l \cap \mathcal{P}$ ,  $0 \leq l \leq k$ . It is shown that  $\text{Ad } u \circ \alpha|_{\mathcal{N}'_l \cap \mathcal{M}}$  is independent from the choice of  $u$ . Thus  $\alpha_l := \text{Ad } u \circ \alpha|_{\mathcal{N}'_l \cap \mathcal{M}}$  is well-defined, i.e.,  $\alpha_l$  does not depend on  $u$  nor  $k$ , and the following definition is justified.

**Definition 2.1** ([13, p.286]) *The Loi invariant  $\Phi(\alpha)$  of  $\alpha$  is defined by  $\Phi(\alpha) = \{\alpha_k\}_{k=0}^{\infty}$ .*

We clarify the behavior of  $\Phi(\alpha)$  by choice of tunnels. Let

$$\mathcal{M} \supset \mathcal{N}_1 \supset \bar{\mathcal{N}}_2 \supset \bar{\mathcal{N}}_3 \cdots, \quad \mathcal{P} \supset \mathcal{Q}_1 \supset \bar{\mathcal{Q}}_2 \supset \bar{\mathcal{Q}}_3 \cdots.$$

be different choice of tunnels. Denote Jones projections for these tunnels by  $e_k^{\bar{\mathcal{N}}} \in \bar{\mathcal{N}}_k$ ,  $e_k^{\bar{\mathcal{Q}}} \in \bar{\mathcal{Q}}_k$ . For any  $k \geq 2$ , there exist unitaries  $v \in \mathcal{N}$ ,  $w \in \mathcal{Q}$  such that  $ve_l^{\bar{\mathcal{N}}}v^* = e_l^{\mathcal{N}}$ ,  $we_l^{\bar{\mathcal{Q}}}w^* = e_l^{\mathcal{Q}}$ .

$we_l^{\bar{Q}}w^* = e_l^{\bar{Q}}$ ,  $0 \leq l \leq k-2$ . This implies  $v\mathcal{N}_lv^* = \bar{\mathcal{N}}_l$ ,  $w\mathcal{Q}_lw^* = \bar{\mathcal{Q}}_l$ ,  $0 \leq l \leq k$ . By this unitary perturbation, we get a Loi invariant  $\{\text{Ad } w \circ \alpha_k \circ \text{Ad } v^*\}_{k=1}^\infty$  for this new tunnel, and this does not depend on the choice of unitaries  $v, w$ . In this sense, the Loi invariant  $\Phi(\alpha)$  is defined canonically for an isomorphism  $\alpha: \mathcal{N} \subset \mathcal{M} \rightarrow \mathcal{Q} \subset \mathcal{P}$ , and it defines an isomorphism of standard invariants. (Later we will explain details of an isomorphism of standard invariants.)

From now on, we assume that involved factors are of type III. Let  $\text{End}_0(\mathcal{M})$  be a set of unital endomorphisms of  $\mathcal{M}$  with finite statistical dimension. For  $\pi_1, \pi_2 \in \text{End}_0(\mathcal{M})$ , we denote the intertwiner space  $(\pi_1, \pi_2) := \{T \in \mathcal{M} \mid T\pi_1(x) = \pi_2(x)T \text{ for all } x \in \mathcal{M}\}$ .

Let  $\rho \in \text{End}_0(\mathcal{M})$ ,  $\sigma \in \text{End}_0(\mathcal{P})$  be endomorphisms such that  $\mathcal{N} = \rho(\mathcal{M})$ ,  $\mathcal{Q} = \sigma(\mathcal{P})$ . Fix standard isometries  $R_\rho \in (\text{id}, \bar{\rho}\rho)$ ,  $\bar{R}_\rho \in (\text{id}, \rho\bar{\rho})$ ,  $R_\sigma \in (\text{id}, \bar{\sigma}\sigma)$  and  $\bar{R}_\sigma \in (\text{id}, \sigma\bar{\sigma})$  such that

$$R_\rho^* \bar{\rho}(\bar{R}_\rho) = \bar{R}_\rho^* \rho(R_\rho) = \frac{1}{d(\rho)}, \quad R_\sigma^* \bar{\sigma}(\bar{R}_\sigma) = \bar{R}_\sigma^* \sigma(R_\sigma) = \frac{1}{d(\sigma)}.$$

Set

$$\mathcal{M} \supset \rho(\mathcal{M}) \supset \rho\bar{\rho}(\mathcal{M}) \supset \rho\bar{\rho}\rho(\mathcal{M}) \supset \cdots = \mathcal{M} \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \mathcal{N}_3 \cdots$$

$$\mathcal{P} \supset \sigma(\mathcal{P}) \supset \sigma\bar{\sigma}(\mathcal{P}) \supset \sigma\bar{\sigma}\sigma(\mathcal{P}) \supset \cdots = \mathcal{P} \supset \mathcal{Q}_1 \supset \mathcal{Q}_2 \supset \mathcal{Q}_3 \cdots$$

Let  $\gamma_\rho = \rho\bar{\rho}$ ,  $\gamma_\sigma = \sigma\bar{\sigma}$  be canonical endomorphisms.

Jones projections  $e_n^\rho \in \mathcal{N}_n$  ( $n = 0, 1, 2, \dots$ ) are given as follows;

$$e_{2k}^\rho = \gamma_\rho^k(\bar{R}_\rho \bar{R}_\rho^*), \quad e_{2k+1}^\rho = \gamma_\rho^k \rho(R_\rho R_\rho^*).$$

Assume there exists an isomorphism  $\alpha: \rho(\mathcal{M}) \subset \mathcal{M} \rightarrow \sigma(\mathcal{P}) \subset \mathcal{P}$ . In the definition of the Loi invariant, there is freedom of choice of a unitary  $u$ . In what follows, we will fix the choice of a unitary by using standard isometries.

Since  $\alpha\rho(\mathcal{M}) = \sigma(\mathcal{P})$ , there exists an isomorphism  $\beta: \mathcal{M} \rightarrow \mathcal{P}$  such that  $\alpha\rho = \sigma\beta$ . Thus  $[\bar{\sigma}\alpha] = [\beta\bar{\rho}]$  holds as sectors, and hence there exists  $u \in U(\mathcal{P})$  such that  $\text{Ad } u \circ \bar{\sigma}\alpha = \beta\bar{\rho}$ . Such unitary  $u$  is not uniquely determined. However we can easily see  $\sigma(u)^*\alpha(\bar{R}_\rho) \in (\text{id}, \sigma\bar{\sigma})$ ,  $u^*\beta(R_\rho) \in (\text{id}, \bar{\sigma}\sigma)$ , and these isometries are also standard solutions for  $\sigma$ ,  $\bar{\sigma}$ . So we can choose  $u \in \mathcal{P}$  in such a way  $\bar{R}_\sigma = \sigma(u)^*\alpha(\bar{R}_\rho) \in (\text{id}, \sigma\bar{\sigma})$ ,  $R_\sigma = u^*\beta(R_\rho) \in (\text{id}, \bar{\sigma}\sigma)$  [20, Proposition 2.2.15]. In what follows, we fix these standard isometries and a unitary  $u$  as above.

Define  $v^{(0)} := 1$  and  $v^{(k+1)} := v^{(k)}(\sigma\bar{\sigma})^k \sigma(u) (= v^{(k)}\gamma_\sigma^k(v^{(1)}))$ . Then  $v^{(k-1)*}v^{(k)} \in \mathcal{Q}_{2k-1}$ , and  $\alpha \circ (\rho\bar{\rho})^k = \text{Ad } v^{(k)} \circ (\sigma\bar{\sigma})^k \circ \alpha$  hold.

Let  $\alpha^{(k)} := \text{Ad } v^{(k)*} \circ \alpha$ . Then  $\alpha^{(k)}(e_l^\rho) = e_l^\rho$ ,  $0 \leq l \leq 2k-1$ , and  $\alpha^{(k)}$  sends  $\mathcal{M} \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_{2k} \supset \mathcal{N}_{2k+1}$  to  $\mathcal{P} \supset \mathcal{Q}_1 \supset \cdots \supset \mathcal{Q}_{2k} \supset \mathcal{Q}_{2k+1}$ . Thus  $\alpha^{(k)}(\mathcal{N}'_l \cap \mathcal{M}) = \mathcal{Q}'_l \cap \mathcal{P}$  for all  $0 \leq l \leq 2k+1$ . Since  $v^{(k-1)*}v^{(k)} \in \mathcal{Q}_{2k-1}$ , we have  $\alpha^{(k)}|_{\mathcal{N}'_{2k-1} \cap \mathcal{M}} = \alpha^{(k-1)}|_{\mathcal{N}'_{2k-1} \cap \mathcal{M}}$ . Thus  $\{\alpha^{(k)}|_{\mathcal{N}'_{2k+1} \cap \mathcal{M}}\}_k$  gives a Loi invariant for  $\alpha$ .

The Loi invariant  $\Phi(\alpha)$  gives an isomorphism of standard invariants of  $\rho(\mathcal{M}) \subset \mathcal{M}$  and  $\sigma(\mathcal{P}) \subset \mathcal{P}$ . Here we clarify the meaning of “an isomorphism of standard invariants” [13, pp. 285].

Let  $A_n^\rho = \mathcal{N}'_n \cap \mathcal{M}$ ,  $B_n^\rho = \mathcal{N}'_n \cap \mathcal{N}$ . These relative commutants are described in terms of intertwiner spaces as follows;

$$A_{2k}^\rho = (\gamma_\rho^k, \gamma_\rho^k), \quad A_{2k+1}^\rho = (\gamma_\rho^k \rho, \gamma_\rho^k \rho),$$

$$B_{2k}^\rho = \rho\left((\bar{\rho}\rho)^{k-1}\bar{\rho}, (\bar{\rho}\rho)^{k-1}\bar{\rho}\right), \quad B_{2k+1}^\rho = \rho\left((\bar{\rho}\rho)^k, (\bar{\rho}\rho)^k\right).$$

We have a canonical inclusion

$$\begin{array}{ccc} B_n^\rho & \subset & B_{n+1}^\rho \\ \cap & & \cap \\ A_n^\rho & \subset & A_{n+1}^\rho \end{array}.$$

Let  $E_n^\rho$  be a conditional expectation  $E_n^\rho: A_n^\rho \rightarrow A_{n-1}^\rho$  defined as follows.

$$E_{2k}(T) = \gamma_\rho^{k-1}\rho(R_\rho^*)T\gamma_\rho^{k-1}\rho(R_\rho), \quad E_{2k+1}(T) = \gamma_\rho^k(\bar{R}_\rho^*)T\gamma_\rho^k(\bar{R}_\rho)$$

It is easy to see  $E_n^\rho|_{B_n^\rho}$  is a conditional expectation  $B_n^\rho \rightarrow B_{n-1}^\rho$ .

Let  $F_n^\rho(T) := \rho(R_\rho^*\bar{\rho}(T)R_\rho)$ ,  $T \in A_n^\rho$ . Then  $F_n^\rho$  is a conditional expectation from  $A_n^\rho$  onto  $B_n^\rho$ .

Via these expectations,

$$\begin{array}{ccc} B_n^\rho & \xrightarrow{E_{n+1}^\rho} & B_{n+1}^\rho \\ F_n^\rho \cap & & \cap F_{n+1}^\rho \\ A_n^\rho & \xrightarrow{E_{n+1}^\rho} & A_{n+1}^\rho \end{array}$$

forms a commuting square.

The standard invariant of  $\rho(\mathcal{M}) \subset \mathcal{M}$  is given by the following nest of finite dimensional algebras

$$\begin{array}{ccccccc} B_1^\rho & \subset & B_2^\rho & \subset & B_3^\rho & \subset & B_4^\rho \cdots \\ \cap & & \cap & & \cap & & \cap \\ A_1^\rho & \subset & A_2^\rho & \subset & A_3^\rho & \subset & A_4^\rho \cdots \end{array}$$

together with conditional expectations and Jones projections.

An isomorphism of standard invariants from  $\rho(\mathcal{M}) \subset \mathcal{M}$  to  $\sigma(\mathcal{P}) \subset \mathcal{P}$  is a family of maps  $\{\alpha_k\}_{k=1}^\infty$ , such that

1.  $\alpha_k$  is an isomorphism from  $A_k^\rho \supset B_k^\rho$  onto  $A_k^\sigma \supset B_k^\sigma$  such that  $F_k^\sigma \alpha_k = \alpha_k F_k^\rho$ .
2.  $\alpha_k|_{A_{k-1}^\rho} = \alpha_{k-1}$ ,  $E_k^\sigma \alpha_k = \alpha_{k-1} E_k^\rho$ .
3.  $\alpha_{k+2}(e_k^\rho) = e_k^\sigma$ . (Note  $e_k^\rho \in A_{k+2}^\rho$ .)

Note that  $\alpha_k$  is an isomorphism of commuting squares

$$\begin{array}{ccc} B_{k-1}^\rho & \subset & B_k^\rho \\ \cap & & \cap \\ A_{k-1}^\rho & \subset & A_k^\rho \end{array} \longrightarrow \begin{array}{ccc} B_{k-1}^\sigma & \subset & B_k^\sigma \\ \cap & & \cap \\ A_{k-1}^\sigma & \subset & A_k^\sigma \end{array}.$$

Here we assume  $\mathcal{N} \subset \mathcal{M}$  and  $\mathcal{Q} \subset \mathcal{P}$  are strongly amenable subfactors of type III<sub>1</sub> in the sense of Popa [22], [21], [23], with identical type II principal graph and type III graph. Then these subfactors are classified by their standard invariants [23] (also see [15].) We state this classification more precisely. There exists a subfactor of type II<sub>1</sub>  $\mathcal{B} \subset \mathcal{A}$ , such that  $\mathcal{B} \subset \mathcal{N}$ ,  $\mathcal{A} \subset \mathcal{M}$ , and a tunnel

$$\mathcal{A} \supset \mathcal{B} \supset \mathcal{B}_1 \supset \cdots,$$

such that  $\mathcal{B} \subset \mathcal{A} = \mathcal{B}^{\text{st}} \subset \mathcal{A}^{\text{st}}$ , and  $\mathcal{N} \subset \mathcal{M} = (\mathcal{B}^{\text{st}} \subset \mathcal{A}^{\text{st}}) \otimes \mathcal{R}_\infty$ . Here  $\mathcal{A}^{\text{st}} = \bigvee_k (\mathcal{B}'_k \cap \mathcal{A})$ ,  $\mathcal{B}^{\text{st}} = \bigvee_k (\mathcal{B}'_k \cap \mathcal{B})$ , and  $\mathcal{R}_\infty$  is the unique injective factor of type III<sub>1</sub> [1], [3].

Let  $\mathcal{T} \subset \mathcal{S}$  be a subfactor of type II<sub>1</sub> which has a same property for  $\mathcal{Q} \subset \mathcal{P}$  as above. If we have an isomorphism  $\theta_0$  on the standard invariants of  $\mathcal{N} \subset \mathcal{M}$  and  $\mathcal{Q} \subset \mathcal{P}$ , then it extends to an isomorphism from  $\mathcal{B}^{\text{st}} \subset \mathcal{A}^{\text{st}}$  onto  $\mathcal{T}^{\text{st}} \subset \mathcal{S}^{\text{st}}$ , and hence to that of  $\mathcal{N} \subset \mathcal{M}$  onto  $\mathcal{Q} \subset \mathcal{P}$ . Moreover, the Loi invariant of this isomorphism coincides with  $\theta_0$ . We summarize this as follows.

**Theorem 2.2** *Let  $\theta_0$  be an isomorphism of standard invariants of  $\mathcal{N} \subset \mathcal{M}$  and  $\mathcal{Q} \subset \mathcal{P}$ . Then there exists an isomorphism  $\theta: \mathcal{N} \subset \mathcal{M} \rightarrow \mathcal{Q} \subset \mathcal{P}$  whose Loi invariant is  $\theta_0$ .*

**Remark.** The above  $\theta$  does not necessary preserve given tunnel for subfactors. We can perturb  $\theta$  so that it preserves any finite length of a tunnel.

### 3 Equivalence and cocycle conjugacy

In [25], Roberts defined an action of a group dual on a von Neumann algebra as a monoidal functor from a representation category of a group to a endomorphism category of a von Neumann algebra. By generalizing his definition, we introduce the notion of a Roberts action of a C\*-tensor category.

**Definition 3.1** Let  $\mathcal{C}$  be a C\*-tensor category, and  $\mathcal{M}$  a factor.

- (1) A Roberts action  $\rho$  of  $\mathcal{C}$  on  $\mathcal{M}$  is a C\*-tensor functor  $\rho: \xi \in \mathcal{C} \mapsto \rho_\xi \in \text{End}_0(\mathcal{M})$  such that  $\rho_{\xi \otimes \eta} = \rho_\xi \rho_\eta$ .
- (2) A Roberts action  $\rho$  of  $\mathcal{C}$  is said to be free if  $\rho$  is a fully faithful functor, i.e.,  $\rho: T \in \text{Mor}(\xi, \eta) \mapsto \rho_T \in (\rho_\xi, \rho_\eta)$  is bijective for any  $\xi, \eta \in \mathcal{C}$ .
- (3) A Roberts action  $\rho$  of  $\mathcal{C}$  is said to be modularly free if it is free and  $\tilde{\rho}_\xi \in \text{End}_0(\tilde{\mathcal{M}})$  is not a modular endomorphism in the sense of [8, Definition 3.1] for  $\xi \not\cong 1_{\mathcal{C}}$ . Here  $\tilde{\mathcal{M}}$  is the core for  $\mathcal{M}$ , and  $\tilde{\rho}_\xi$  is the canonical extension of  $\rho_\xi$  in the sense of [8, Theorem 2.4].

**Remark.** (1) In the rest of this paper, we use a same letter  $T \in (\rho_\xi, \rho_\eta)$  for an intertwiner  $T \in \text{Mor}(\xi, \eta)$  to simplify notation.

(2) When  $\mathcal{M}$  is a factor of type III<sub>1</sub>, the modular freeness of  $\rho$  is equivalent to the full faithfulness of a functor  $\xi \in \mathcal{C} \mapsto \tilde{\rho}_\xi \in \text{End}_0(\tilde{\mathcal{M}})$ . In general, the modular freeness of  $\rho$  is equivalent to  $(\tilde{\rho}_\xi, \tilde{\rho}_\eta) = Z(\tilde{\mathcal{M}}) \otimes (\rho_\xi, \rho_\eta)$  for  $\xi, \eta \in \mathcal{C}$ . (Here we have  $Z(\tilde{\mathcal{M}}), (\rho_\xi, \rho_\eta) \subset (\tilde{\rho}_\xi, \tilde{\rho}_\eta)$  and  $Z(\tilde{\mathcal{M}}) \cap (\rho_\xi, \rho_\eta) = \mathbb{C}$ . Hence we can regard  $Z(\tilde{\mathcal{M}}) \otimes (\rho_\xi, \rho_\eta) \subset (\tilde{\rho}_\xi, \tilde{\rho}_\eta)$ .)

(3) When  $\mathcal{M}$  is injective, the modular freeness is equivalent to the central freeness [12, Theorem 1], [18, Theorem 4.12].

Let  $\mathcal{C}, \mathcal{D}$  be unitarily equivalent C\*-tensor categories, and  $(F, L)$  be a tensor equivalence functor of  $\mathcal{C}$  and  $\mathcal{D}$ . Namely,  $F$  is a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$ , which gives surjective isomorphisms on intertwiner spaces, and  $L$  is a natural unitary equivalence of tensor product  $L(\xi, \eta) \in (F(\xi) \otimes F(\eta), F(\xi \otimes \eta))$ . We assume  $F(1_{\mathcal{C}}) = 1_{\mathcal{D}}$ , and  $L(1_{\mathcal{C}}, \xi) = L(\xi, 1_{\mathcal{C}}) = 1$ . In the following, we omit  $\otimes$ , and denote  $\xi \otimes \eta$  by  $\xi\eta$  for simplicity.

Let  $\rho$  (resp.  $\sigma$ ) be a modularly free Roberts action of  $\mathcal{C}$  on  $\mathcal{M}$  (resp.  $\mathcal{D}$  on  $\mathcal{P}$ ), where  $\mathcal{M}$  and  $\mathcal{P}$  are factors of type III<sub>1</sub>. Such actions exist by the result of Hayashi-Yamagami [4, Theorem 7.6] together with a simple trick by making tensor product with an injective factor of type III<sub>1</sub>, if  $\mathcal{C}$  and  $\mathcal{D}$  are amenable and factors are injective.

By the remark (1) after Definition 3.1, an element of  $(\sigma_{F(\xi)}, \sigma_{F(\eta)})$  can be expressed by  $F(T)$ ,  $T \in (\rho_\xi, \rho_\eta)$ , and  $L(\xi, \eta)$  can be regarded as a unitary in  $(\sigma_{F(\xi)}\sigma_{F(\eta)}, \sigma_{F(\xi\eta)})$ .

We extend  $L(\xi, \eta)$  to  $L(\xi_1, \dots, \xi_n) \in (\sigma_{F(\xi_1)} \cdots \sigma_{F(\xi_n)}, \sigma_{F(\xi_1 \cdots \xi_n)})$  in the canonical way. We use the notation  $L(\xi) = 1$  for  $\xi \in \mathcal{C}$ . For  $T \in (\rho_{\xi_1} \cdots \rho_{\xi_n}, \rho_{\eta_1} \cdots \rho_{\eta_m})$ , let

$$\theta(T) := L(\eta_1, \dots, \eta_m)^* F(T) L(\xi_1, \dots, \xi_n) \in (\sigma_{F(\xi_1)} \cdots \sigma_{F(\xi_n)}, \sigma_{F(\eta_1)} \cdots \sigma_{F(\eta_m)}).$$

Although we should use the notation like  $\theta_{\xi_1, \dots, \xi_n}^{\eta_1, \dots, \eta_m}(T)$  for completeness, we simply denote by  $\theta(T)$ .

**Lemma 3.2** *We have the followings.*

(1) *We have  $\theta(ST) = \theta(S)\theta(T)$  and  $\theta(T)^* = \theta(T^*)$  for  $T \in (\rho_{\xi_1} \cdots \rho_{\xi_n}, \rho_{\eta_1} \cdots \rho_{\eta_m})$ ,  $S \in (\rho_{\eta_1} \cdots \rho_{\eta_m}, \rho_{\zeta_1} \cdots \rho_{\zeta_l})$ .*

(2)  *$\theta(T)$  is well-defined in the following sense. If we regard  $T \in (\rho_{\xi_1} \cdots \rho_{\xi_n}, \rho_{\eta_1} \cdots \rho_{\eta_m})$  as an element in  $(\rho_{\xi_1} \cdots \rho_{\xi_n} \rho_\zeta, \rho_{\eta_1} \cdots \rho_{\eta_m} \rho_\zeta)$ , then we have*

$$L(\eta_1, \dots, \eta_m)^* F(T) L(\xi_1, \dots, \xi_n) = L(\eta_1, \dots, \eta_m, \zeta)^* F(T) L(\xi_1, \dots, \xi_n, \zeta).$$

(3)  *$\sigma_{F(\zeta)}\theta(T) = \theta(\rho_\zeta(T))$  for  $T \in (\rho_{\xi_1} \cdots \rho_{\xi_n}, \rho_{\eta_1} \cdots \rho_{\eta_m})$ .*

**Proof.** (1) It is easy to see  $\theta(T)^* = \theta(T^*)$ . We can verify  $\theta(TS) = \theta(T)\theta(S)$  as follows.

$$\begin{aligned} \theta(S)\theta(T) &= L(\zeta_1, \dots, \zeta_l)^* F(S) L(\eta_1, \dots, \eta_m) L(\eta_1, \dots, \eta_m)^* F(T) L(\xi_1, \dots, \xi_n) \\ &= L(\zeta_1, \dots, \zeta_l)^* F(S) F(T) L(\xi_1, \dots, \xi_n) \\ &= \theta(ST). \end{aligned}$$

(2) Note  $L(\xi_1, \dots, \xi_n, \zeta) = L((\xi_1 \cdots \xi_n), \zeta) L(\xi_1, \dots, \xi_n)$ . Then

$$\begin{aligned} &L(\eta_1, \dots, \eta_m, \zeta)^* F(T) L(\xi_1, \dots, \xi_n, \zeta) \\ &= L(\eta_1, \dots, \eta_m)^* L((\eta_1 \cdots \eta_m), \zeta)^* F(T) L((\xi_1 \cdots \xi_n), \zeta) L(\xi_1, \dots, \xi_n) \\ &= L(\eta_1, \dots, \eta_m)^* L((\eta_1 \cdots \eta_m), \zeta)^* L((\eta_1 \cdots \eta_m), \zeta) F(T) L(\xi_1, \dots, \xi_n) \text{ (by naturality of } L) \\ &= L(\eta_1, \dots, \eta_m)^* F(T) L(\xi_1, \dots, \xi_n). \end{aligned}$$

(3) Note  $L(\zeta, \xi_1, \dots, \xi_n) = L(\zeta, (\xi_1 \cdots \xi_n)) \sigma_{F(\zeta)}(L(\xi_1, \dots, \xi_n))$ . Then

$$\begin{aligned} &\theta(\rho_\zeta(T)) \\ &= L(\zeta, \eta_1, \dots, \eta_m)^* F(\rho_\zeta(T)) L(\zeta, \xi_1, \dots, \xi_n) \\ &= \sigma_{F(\zeta)}(L(\eta_1, \dots, \eta_m)^* L(\zeta, (\eta_1 \cdots \eta_m))^* F(\rho_\zeta(T)) L(\zeta, (\xi_1 \cdots \xi_n)) \sigma_{F(\zeta)}(L(\xi_1, \dots, \xi_n))) \\ &= \sigma_{F(\zeta)}(L(\eta_1, \dots, \eta_m)^* L(\zeta, (\eta_1 \cdots \eta_m))^* L(\zeta, (\eta_1 \cdots \eta_m)) \sigma_{F(\zeta)}(F(T)) \sigma_{F(\zeta)}(L(\xi_1, \dots, \xi_n))) \\ &= \sigma_{F(\zeta)}(L(\eta_1, \dots, \eta_m)^* \sigma_{F(\zeta)}(F(T)) \sigma_{F(\zeta)} L(\xi_1, \dots, \xi_n)) \\ &= \sigma_{F(\zeta)}(\theta(T)). \end{aligned}$$

Here the third equality is due to the naturality of  $L$ . □

We present the definition of cocycle conjugacy of two Roberts actions of  $C^*$ -tensor category.

**Definition 3.3** Let  $\mathcal{C}$  and  $\mathcal{D}$  be unitarily equivalent  $C^*$ -tensor categories with tensor equivalence  $(F, L)$ . Let  $\rho$  (resp.  $\sigma$ ) be a Roberts action of  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) on a factor  $\mathcal{M}$  (resp.  $\mathcal{P}$ ). We say  $\rho$  and  $\sigma$  are cocycle conjugate if there exist an isomorphism  $\alpha: \mathcal{M} \rightarrow \mathcal{P}$  and a unitary  $\mathcal{E}(\xi) \in \mathcal{P}$  for every  $\xi \in \mathcal{C}$  such that

- (i)  $\mathcal{E}(1_{\mathcal{C}}) = 1$ ,
- (ii)  $\mathcal{E}(\xi) \in (\sigma_{F(\xi)}\alpha, \alpha\rho_{\xi})$ ,
- (iii)  $\mathcal{E}(\xi)\sigma_{F(\xi)}(\mathcal{E}(\eta)) = \mathcal{E}(\xi\eta)L(\xi, \eta)$ ,
- (iv)  $\alpha(T)\mathcal{E}(\xi) = \mathcal{E}(\eta)F(T)$  for  $T \in \text{Mor}(\xi, \eta)$ . In particular,  $\mathcal{E}(\xi) = \sum \alpha(T_i)\mathcal{E}(\xi_i)F(T_i)^*$  holds for a (not necessary irreducible) decomposition  $\rho_{\xi}(x) = \sum_i T_i\rho_{\xi_i}(x)T_i^*$ .

**Remark.** In Definition 3.3, we do not assume the simplicity of objects.

Assume  $\mathcal{C}$  and  $\mathcal{D}$  are finitely generated. Let  $\mathcal{C}_0 \subset \text{Irr}(\mathcal{C})$  be a (finite) generator of  $\mathcal{C}$ . We assume that  $1_{\mathcal{C}} \in \mathcal{C}_0$  and  $\mathcal{C}_0$  is closed under conjugation.

Let  $\pi := \bigoplus_{\xi \in \mathcal{C}_0} \xi$ , and consider subfactors  $\rho_{\pi}(\mathcal{M}) \subset \mathcal{M}$  and  $\sigma_{F(\pi)}(\mathcal{P}) \subset \mathcal{P}$ . (Note  $\rho_{\pi}$  is self-conjugate.) Set  $\gamma_{\rho} := \rho_{\pi}\rho_{\pi}$ ,  $\gamma_{\sigma} := \sigma_{F(\pi)}\sigma_{F(\pi)}$ .

We fix standard isometries  $R_{\rho} \in (\text{id}, \bar{\rho}_{\pi}\rho_{\pi})$ ,  $\bar{R}_{\rho} \in (\text{id}, \rho_{\pi}\bar{\rho}_{\pi})$ ,  $R_{\sigma} \in (\text{id}, \bar{\sigma}_{F(\pi)}\sigma_{F(\pi)})$ ,  $\bar{R}_{\sigma} \in (\text{id}, \sigma_{F(\pi)}\bar{\sigma}_{F(\pi)})$  so that  $\theta(R_{\rho}) = R_{\sigma}$ ,  $\theta(\bar{R}_{\rho}) = \bar{R}_{\sigma}$ .

Standard invariants of these subfactors depends only on  $\mathcal{C}$ ,  $\mathcal{D}$  and not on actions  $\rho$ ,  $\sigma$ . In fact, by Lemma 3.2 we can see that an isomorphism between standard invariants is given by

$$\theta: (\rho_{\pi}^n, \rho_{\pi}^n) \rightarrow (\sigma_{F(\pi)}^n, \sigma_{F(\pi)}^n),$$

and hence by an equivalence  $(F, L)$ . Since  $\rho$  and  $\sigma$  are modularly free, these subfactors have same type II principal graphs and type III graphs by [7, Theorem 3.5].

Let us assume  $\mathcal{C}$  and  $\mathcal{D}$  are strongly amenable, i.e., fusion algebras  $\mathbb{C}[\text{Irr}(\mathcal{C})]$ ,  $\mathbb{C}[\text{Irr}(\mathcal{D})]$  are strongly amenable in the sense of [5, Definition 6.4]. By [5, Theorem 4.8], this is equivalent to the strong amenability of standard invariants of  $\rho_{\pi}(\mathcal{M}) \subset \mathcal{M}$  and  $\sigma_{F(\pi)}(\mathcal{P}) \subset \mathcal{P}$  in the sense of Popa [22], [21], [23].

We state the main theorem of this paper.

**Theorem 3.4** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be unitarily equivalent finitely generated strongly amenable  $C^*$ -tensor categories with tensor equivalence  $(F, L)$ . Let  $\rho$  (resp.  $\sigma$ ) be a modularly free Roberts action of  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) on an injective factor of type  $\text{III}_1$   $\mathcal{M}$  (resp.  $\mathcal{P}$ ). Then  $\rho$  and  $\sigma$  are cocycle conjugate.*

The following corollary immediately follows if we put  $\mathcal{C} = \mathcal{D}$ ,  $\mathcal{M} = \mathcal{P}$ , and  $(F, L)$  as an identity functor.

**Corollary 3.5** *Let  $\mathcal{C}$  be a finitely generated strongly amenable  $C^*$ -tensor category. Let  $\rho$  and  $\sigma$  be modularly free Roberts actions of  $\mathcal{C}$  on the injective factor  $\mathcal{M}$  of type  $\text{III}_1$ . Then there exist  $\alpha \in \text{Aut}(\mathcal{M})$  and a unitary  $\mathcal{E}(\xi)$  for every  $\xi \in \mathcal{C}$  such that*

- (i)  $\text{Ad } \mathcal{E}(\xi) \circ \sigma_{\xi} = \alpha \circ \rho_{\xi} \circ \alpha^{-1}$  for every  $\xi \in \mathcal{C}$ .
- (ii)  $\mathcal{E}(\xi)\sigma_{F(\xi)}(\mathcal{E}(\eta)) = \mathcal{E}(\xi\eta)$ .
- (iii)  $\alpha(T)\mathcal{E}(\xi) = \mathcal{E}(\eta)F(T)$  for  $T \in \text{Mor}(\xi, \eta)$ .

**Proof of Theorem 3.4.** The proof is a simple modification of proof of [14, Theorem 2.1].

By Theorem 2.2, there exists an isomorphism  $\alpha: \rho_\pi(\mathcal{M}) \subset \mathcal{M} \rightarrow \sigma_{F(\pi)}(\mathcal{P}) \subset \mathcal{P}$  whose Loi invariant is  $\theta$ . Then we will choose  $u \in \mathcal{P}$  so that  $\bar{R}_\sigma = \sigma(u^*)\alpha(\bar{R}_\rho)$  as explained in §2.

Let us summarize facts which are used in the proof.

$$v^{(1)} = \sigma(u), \quad v^{(n+1)} = v^{(n)}\gamma_\sigma^n(v^{(1)}) = v^{(1)}\gamma_\sigma(v^{(n)}),$$

$$\alpha \circ \gamma_\rho^n = \text{Ad } v^{(n)} \circ \gamma_\sigma^n \circ \alpha, \quad \alpha^{(n)} = \text{Ad } v^{(n)*} \circ \alpha,$$

$$\alpha^{(n)} = \theta \text{ on } (\gamma_\rho^n, \gamma_\rho^n),$$

$$\theta(\bar{R}_\rho) = \bar{R}_\sigma = \sigma(u^*)\alpha(\bar{R}_\rho) = v^{(1)*}\alpha(\bar{R}_\rho).$$

Let  $\xi \in \mathcal{C}$ . Since  $\mathcal{C}_0$  generates  $\mathcal{C}$ , there exist  $n \in \mathbb{N}$  and an isometry  $T \in (\rho_\xi, \gamma_\rho^n)$ .

Define  $W_T = \alpha(T^*)v^{(n)}\theta(T)$ . We will verify that  $W_T$  is a desired unitary  $\mathcal{E}(\xi)$ .

Our first task is to show that  $W_T$  is a unitary and independent from the choice of  $T$  and  $n$ .

First, we show  $W_T = W_S$  for  $T, S \in (\rho_\xi, \gamma_\rho^n)$ . We compute  $W_T^*W_S = 1$ . We have

$$W_T^*W_S = \theta(T^*)v^{(n)*}\alpha(T)\alpha(S^*)v^{(n)}\theta(S) = \theta(T^*)\alpha^{(n)}(TS^*)\theta(S).$$

Here  $TS^* \in (\gamma_\rho^n, \gamma_\rho^n)$ , and  $\alpha^{(n)}$  on  $(\gamma_\rho^n, \gamma_\rho^n)$  is given by  $\theta$ . Hence

$$\theta(T^*)\alpha^{(n)}(TS^*)\theta(S) = \theta(T^*)\theta(TS^*)\theta(S) = 1$$

holds. Similarly we get  $W_SW_T^* = 1$ . This shows that  $W_T$  is a unitary (put  $T = S$ ), and  $W_T = W_S$ .

Next we show that  $W_T$  does not depend on  $n$ . The following proof is suggested by R. Tomatsu, which is much simpler than author's original proof. Let  $T \in (\rho_\xi, \gamma_\rho^n)$  be an isometry. Then  $\bar{R}_\rho T \in (\rho_\xi, \gamma_\rho^{n+1})$ .

$$\begin{aligned} W_{\bar{R}_\rho T} &= \alpha(T^*\bar{R}_\rho^*)v^{(n+1)}\theta(\bar{R}_\rho T) \\ &= \alpha(T^*\bar{R}_\rho^*)v^{(1)}\gamma_\sigma(v^{(n)})\theta(\bar{R}_\rho T) \\ &= \alpha(T^*)\bar{R}_\sigma^*\gamma_\sigma(v^{(n)})\theta(\bar{R}_\rho)\theta(T) \\ &= \alpha(T^*)v^{(n)}\bar{R}_\sigma^*\bar{R}_\sigma\theta(T) \\ &= \alpha(T^*)v^{(n)}\theta(T) \\ &= W_T. \end{aligned}$$

Thus  $\mathcal{E}(\xi) := W_T$  is well-defined.

We will show that  $\mathcal{E}(\xi)$  satisfies the condition (i)–(iv) in Definition 3.3.

(i) Since  $\bar{R}_\rho \in (\text{id}, \gamma_\rho)$ ,

$$\mathcal{E}(1_{\mathcal{C}}) = W_{\bar{R}_\rho} = \alpha(\bar{R}_\rho^*)v^{(1)}\theta(\bar{R}_\rho) = \bar{R}_\sigma^*\bar{R}_\sigma = 1.$$



(ii) Choose  $n \in \mathbb{N}$  and an isometry  $T \in (\rho_\xi, \gamma_\rho^n)$ . Note  $\theta(T) \in (\sigma_{F(\xi)}, \gamma_\sigma^n)$ . Then  $\mathcal{E}(\xi) = W_T \in (\sigma_{F(\xi)}\alpha, \alpha\rho_\xi)$  is verified as follows;

$$\begin{aligned}\alpha\rho_\xi(x)W_T &= \alpha\rho_\xi(x)\alpha(T^*)v^{(n)}\theta(T) \\ &= \alpha(T^*)\alpha\gamma_\rho^n(x)v^{(n)}\theta(T) \\ &= \alpha(T^*)v^{(n)}\gamma_\sigma^n\alpha(x)\theta(T) \\ &= \alpha(T^*)v^{(n)}\theta(T)\sigma_{F(\xi)}\alpha(x) \\ &= W_T\sigma_{F(\xi)}\alpha(x).\end{aligned}$$

To prove (iii) and (iv), we claim the following.

**Claim.** We have

$$\mathcal{E}(\xi)\sigma_{F(\xi)}(\mathcal{E}(\eta))\theta(T) = \alpha(T)\mathcal{E}(\zeta)$$

for  $\xi, \eta, \zeta \in \mathcal{C}$  and an isometry  $T \in (\rho_\zeta, \rho_\xi\rho_\eta)$ .

Take  $n, m \in \mathbb{N}$  and isometries  $S_\xi \in (\rho_\xi, \gamma_\rho^n)$ ,  $S_\eta \in (\rho_\eta, \gamma_\rho^m)$ . Then  $S_\zeta := S_\xi\rho_\xi(S_\eta)T \in (\rho_\zeta, \gamma_\rho^{n+m})$  is an isometry, and  $S_\zeta S_\zeta^* \in (\gamma_\rho^{n+m}, \gamma_\rho^{n+m})$ . So we have  $\alpha^{(n+m)}(S_\zeta S_\zeta^*) = \theta(S_\zeta S_\zeta^*)$ .

We show  $W_{S_\xi}\sigma_{F(\xi)}(W_{S_\eta})\theta(T) = \alpha(T)W_{S_\zeta}$ ;

$$\begin{aligned}W_{S_\xi}\sigma_{F(\xi)}(W_{S_\eta})\theta(T) &= W_{S_\xi}\sigma_{F(\xi)}\alpha(S_\eta^*)\sigma_{F(\xi)}(v^{(m)})\sigma_{F(\xi)}(\theta(S_\eta))\theta(T) \\ &= \alpha\rho_\xi(S_\eta^*)W_{S_\xi}\sigma_{F(\xi)}(v^{(m)})\sigma_{F(\xi)}(\theta(S_\eta))\theta(T) \\ &= \alpha\rho_\xi(S_\eta^*)\alpha(S_\xi^*)v^{(n)}\theta(S_\xi)\sigma_{F(\xi)}(v^{(m)})\sigma_{F(\xi)}(\theta(S_\eta))\theta(T) \\ &= \alpha\rho_\xi(S_\eta^*)\alpha(S_\xi^*)v^{(n)}\gamma_\sigma^n(v^{(m)})\theta(S_\xi)\theta(\rho_\xi(S_\eta))\theta(T) \quad (\text{by Lemma 3.2(2)}) \\ &= \alpha(\rho_\xi(S_\eta^*)S_\xi^*)v^{(n+m)}\theta(S_\xi\rho_\xi(S_\eta)T) \\ &= v^{(n+m)}\alpha^{(n+m)}(\rho_\xi(S_\eta^*)S_\xi^*)\theta(S_\zeta) \\ &= v^{(n+m)}\alpha^{(n+m)}(\rho_\xi(S_\eta^*)S_\xi^*)\theta(S_\zeta S_\zeta^* S_\zeta) \quad (\text{since } S_\zeta \text{ is an isometry}) \\ &= v^{(n+m)}\alpha^{(n+m)}(\rho_\xi(S_\eta^*)S_\xi^* S_\zeta S_\zeta^*)\theta(S_\zeta) \\ &= v^{(n+m)}\alpha^{(n+m)}(TS_\zeta^*)\theta(S_\zeta) \\ &= \alpha(T)\alpha(S_\zeta^*)v^{(n+m)}\theta(S_\zeta) \\ &= \alpha(T)W_{S_\zeta}.\end{aligned}$$

Thus Claim has been verified. We continue the proof of Theorem 3.4.

(iii) Put  $\zeta = \xi\eta$ , and  $T = 1 \in (\rho_{\xi\eta}, \rho_\xi\rho_\eta)$  in Claim. Note  $\theta(T) = L(\xi, \eta)^*F(1)L(\xi\eta) = L(\xi, \eta)^*$  in this case. Then we immediately get (iii).

(iv) If  $(\xi, \eta) \neq 0$ , then there exists  $\zeta \in \mathcal{C}$  such that  $\zeta \prec \xi$ ,  $\zeta \prec \eta$ . Take isometries  $T \in \text{Mor}(\zeta, \xi)$  and  $S \in \text{Mor}(\zeta, \eta)$ . Then  $ST^* \in \text{Mor}(\xi, \eta)$ . We only have to show (iv) for  $ST^*$ , since every element in  $\text{Mor}(\xi, \eta)$  is a linear span of intertwiners of such form.

By Claim, we have

$$\mathcal{E}(\xi)\theta(T) = \alpha(T)\mathcal{E}(\zeta), \quad \mathcal{E}(\eta)\theta(S) = \alpha(S)\mathcal{E}(\zeta).$$

(Put  $\xi = 1_e$  or  $\eta = 1_e$  in Claim.) Then

$$\alpha(ST^*)\mathcal{E}(\xi) = \alpha(S)\mathcal{E}(\zeta)\theta(T)^* = \mathcal{E}(\eta)\theta(ST^*) = \mathcal{E}(\eta)F(ST^*).$$

Note  $\theta(ST^*) = L(\eta)^*F(ST^*)L(\xi) = F(ST^*)$ .

□

**Remark.** (1) Even when  $\mathcal{M}$  and  $\mathcal{P}$  are general AFD type III factors, Theorem 3.4 is true provided that  $\rho_\xi$  and  $\sigma_\xi$  have trivial Connes-Takesaki modules in the sense of [8] thanks to the classification results [13, Theorem 6.1], [26, Theorem 4.2] of subfactors of type  $\text{III}_\lambda$ ,  $\lambda \neq 1$ .

(2) We can easily see that the statement in Claim is equivalent to the conditions (iii), (iv) in Definition 3.3 under the condition  $\mathcal{E}(1_{\mathcal{C}}) = 1$ .

We present typical applications of our main theorem.

**Example 3.6** When  $\mathcal{C}$  is a finite  $C^*$ -tensor category, a free Roberts action of  $\mathcal{C}$  is automatically modularly free. Thus Theorem 3.4 is a generalization of [9, Theorem 2.2].

Next application is classification of group actions.

**Example 3.7** Let  $\Gamma$  be a discrete group. Then a Roberts action of  $\text{Vec}_\Gamma$  is nothing but a usual action of  $\Gamma$ . If  $\Gamma$  is a finitely generated strongly amenable group, then the main theorem implies that two centrally free actions of  $\Gamma$  on the injective factor of type  $\text{III}_1$  are cocycle conjugate. (Note that complete classification has been already obtained in [12], [11], [16] by different methods.)

It is worth mentioning the following remark. If we consider only an isomorphism between locally trivial subfactors, then we get only outer conjugacy of actions. To obtain cocycle conjugacy, one must examine the Loi invariant carefully.

Other interesting application of the main theorem is Popa-Wasserman's theory on classification of compact Lie group actions.

**Example 3.8 ([24])** Let  $\mathbb{G}$  be a compact Kac algebra such that  $\text{Rep}(\mathbb{G})$  is strongly amenable, and finitely generated. Typical examples of such compact Kac algebras come from compact Lie groups [22], [5]. Then the main theorem implies the uniqueness of centrally free Roberts actions of  $\text{Rep}(\mathbb{G})$  on the injective factor of type  $\text{III}_1$ . Let  $\pi$  be a generator of  $\text{Rep}(\mathbb{G})$ , and set  $\tilde{\pi} := \text{id} \oplus \pi \oplus \bar{\pi}$ . Let  $\alpha$  be a modularly free action of  $\text{Rep}(\mathbb{G})$  on an injective factor  $\mathcal{M}$  of type  $\text{III}_1$  and  $\hat{\alpha}$  a dual action of  $\mathbb{G}$  on  $\mathcal{N} = \mathcal{M} \rtimes_\alpha \hat{\mathbb{G}}$ . Consider a Wasserman subfactor  $\mathcal{N}^{\hat{\alpha}} \subset (\mathcal{N} \otimes M_{d\tilde{\pi}}(\mathbb{C}))^{\hat{\alpha} \otimes \text{Ad } \tilde{\pi}}$ . This is isomorphic to the dual inclusion of  $\alpha_{\tilde{\pi}}(\mathcal{M}) \subset \mathcal{M}$ . Thus main theorem implies  $\alpha$  is unique up to cocycle conjugacy, and hence  $\hat{\alpha}$  is unique up to conjugacy.

Note that centrally free actions of  $\text{Rep}(\mathbb{G})$  on the injective factor of type  $\text{III}_1$  is classified completely in [19, Theorem 7.16] for a general coamenable Kac algebra  $\mathbb{G}$  by a different method.

We close this paper with the following comments.

As long as we apply classification results of subfactors, it seems to be impossible to remove the assumption that tensor categories are finitely generated. Thus it is desirable to develop methods used in [17], [19] to generalize Theorem 3.4 for arbitrary amenable  $C^*$ -tensor categories.

**Problem.** For amenable  $C^*$ -tensor categories, show the uniqueness of modularly free Roberts actions on the injective factor of type  $\text{III}_1$ .

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